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STABILITY AND CONTROLLABILITY IN PROPORTIONAL NAVIGATION*

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A discussion is presented of the possibility of applying the methods developed in /1, 2/ to the case of arbitrary values of the navigation constant, thus permitting, in particular, treatment of the case in which this constant is less than unity (weak regulation), as well as cases in which the value of the constant lies between 1 and 2. It is shown that by considering the motion on a Riemann surface one can avoid the complications due to the fact that the right-hand sides of the differential equations are multivalued functions. Partition of the parameter plane, as developed previously in /1/, is extended to the general case, thus making it possible in principle to carry out a qualitative investigation of the nature of the motion (including stability and controllability), without actually solving the equations. An example is presented.

Previous investigations of stability and controllability in proportional navigation have frequently been based on the linearized theory (see, e.g., /2/). However, whereas this approach can be justified in the context of manoeuvrability, since the regulators keep the appropriate angles sufficiently small, linearization of the fundamental equations is unjustifiable. During the motion the angle may vary by tens and hundreds of degrees (in some cases the moving point describes even more than one revolution about the origin), and linearization may lead to quite incorrect conclusions.

It will be shown below that stability and controllability can be investigated for arbitrary values of the navigation constant, using the exact equations of motion of the centre of mass, with linearization applied only for rotation about the centre of mass.

1. Fundamental equations. If the pursued point is moving in a straight line at constant velocity, the equations of relative motion of the centre of mass of the pursuing body have the form /1/

$$a^{\cdot} = v_s \cos\eta - v\cos\gamma, \quad a\eta^{\cdot} = v\sin\gamma - v_s\sin\eta, \quad \eta = \psi + \gamma, \quad \psi = b\eta^{\cdot} \tag{1.1}$$

Here a is the relative distance, η the angle of inclination of the slighting line, ψ the angle of inclination of the absolute velocity, γ the lead angle, v the velocity of the centre of mass of the pursuing body, v_s the velocity of the pursued point, and b the navigation constant (Fig.1).

If $b \neq 1$ (the case $b = 1$ corresponds to classical pursuit or pursuit with anticipation and will not be considered here) Eqs.(1.1) can be reduced to the form /1/

$$\begin{gathered} a^{\cdot} = v_1 [\cos\eta - p\cos(b-1)(\eta - \varepsilon_0)] = v_s F(\eta) \\ a\eta^{\cdot} = v_1 [\sin\eta + p\sin(b-1)(\eta-\varepsilon_0)] = -v_s f(\eta) \\ \varepsilon_0 = \eta_0 + \frac{\gamma_0}{b-1} = \frac{b}{b-1}\eta_0 - \frac{1}{b-1}\psi_0, \quad p = \frac{v}{v_1} \end{gathered} \tag{1.2}$$

where ε_0, p are new parameters, with ε_0 depending on the initial conditions.

The quantities a and η in Eqs.(1.2) may be treated as polar coordinates of a point B in relative motion relative to a point A, the pole being placed at A and the polar axis pointing in the opposite direction to the vector v_s (if it pointed in the same direction as v_s, the polar coordinates would be $(a, \pi + \eta)$). Eliminating t from Eqs.(1.2), we obtain the equation of the relative trajectory:

$$\frac{1}{a}\frac{da}{d\eta} = -\frac{\cos\eta - p\cos(b-1)(\eta-\varepsilon_0)}{\sin\eta + p\sin(b-1)(\eta-\varepsilon_0)} = -\frac{F(\eta)}{f(\eta)} \tag{1.3}$$

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When one passes from the system of Eqs.(1.1) to Eq.(1.3), the initial data are partially incorporated in the differential equation itself as parameters (through ε_0). Here and below, therefore, the full notation for the right-hand sides, auxiliary functions and solutions should be $F(\eta, \varepsilon_0)$, $L(\eta, \varepsilon_0)$, $a(\eta, \varepsilon_0, a_0, t_0)$ and so on.

After finding the equation of the trajectory, one determines the time of the motion from the second equation of (1.2);

$$t = -\int_{\eta_0}^{\eta} \frac{a(\eta)\, d\eta}{v_s f(\eta)} \tag{1.4}$$

Eqs.(1.2)-(1.4) enable one to carry out a qualitative analysis of the trajectory and the motion. When this is done it is more convenient to write Eq.(1.3) as $\operatorname{tg} \nu = -f(\eta)/F(\eta)$, where ν is the angle between the relative trajectory and the radius-vector (the positive direction of the trajectory is assumed to be the direction of increasing η; see Fig.2). Hence

$$\frac{d\nu}{d\eta} = \frac{L(\eta)}{G^2(\eta)}; \; L(\eta) = \frac{b-2}{2} G^2(\eta) + \frac{b}{2}(p^2 - 1), \quad G^2(\eta) = f^2(\eta) + F^2(\eta) \tag{1.5}$$

Eqs.(1.2) and (1.5) also yield the curvature of the relative trajectory:

$$K = |f(\eta)|\, \Omega(\eta) / (aG^3(\eta)) \tag{1.6}$$
$$\Omega(\eta) = L(\eta) + G^2(\eta) = pb\{p - \cos[b\eta - (b-1)(\eta - \varepsilon_0)]\}$$

The sign of the curvature (which is the same as the sign of $\Omega(\eta)$) is defined here relative to the pole, i.e., $K > 0$ or $K < 0$ according as the trajectory is concave toward the pole (lies nearer the latter than the tangent) or convex toward the pole.

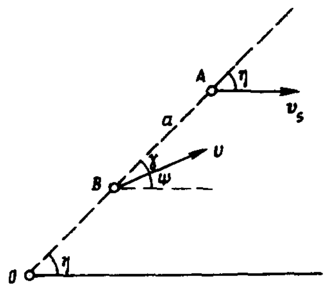

Fig.1

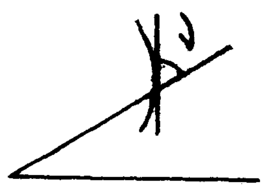
Fig.2

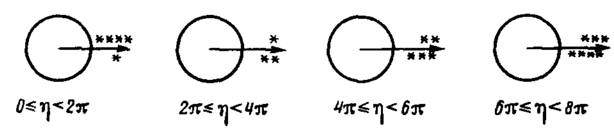

Fig.3

2. ***The shape of the trajectories and the nature of the motion.*** It is obvious from Eqs. (1.2) that the functions $f(\eta)$ and $F(\eta)$ uniquely define the form of the trajectory and the nature of the motion (the factor v_s affects the scale only). However, a qualitative investigation of the motion for fractional values of the navigation constant (unlike the case of integral b considered in /1/) encounters difficulties, due to the fact that $f(\eta)$ and $F(\eta)$ are not always single-valued functions of the position of the point. Indeed, as the point moves the angle η may leave the interval $[0, 2\pi]$, and if b is a fraction the values of $f(\eta)$ and $F(\eta)$ at η and $\eta + 2\pi$ need not coincide. Thus, to compute the right-hand sides of Eqs.(1.2) (or of the auxiliary functions) at any point, one must survey the "prehistory" of the motion.

To explain the method proposed here, an example of a trajectory will be considered in Sect.6.

Another approach is to assume that with each passage through the ray $\eta = 0\,(2\pi)$ the polar angle changed in a jumpwise fashion by 2π; but this would introduce discontinuities which are quite unjustified by physical reality. For fractional values of b, therefore, it is more convenient to replace the real plane of the motion by a Riemann surface and to consider the motion on the surface. The first sheet of the surface will necessarily be the plane of the real motion; the rays $\eta = 0,\ 2\pi,\ 4\pi, \ldots$ are cuts along which one can pass from one sheet to the next. The number of sheets will be equal to q if $b = l/q$ is rational (and the ray $\eta = 2\pi q$ will then coincide with the ray $\eta = 0$), infinite if b is irrational. Of course, the true trajectory is obtained by projecting the Riemann surface onto its first sheet.

Fig.3 illustrates the Riemann surface and the correspondence between the edges of the cuts for $q = 4$ showing all four sheets; the asterisks indicate coinciding edges of the different sheets.

This approach is also extremely convenient in considerations of the differential Eq.(1.3) of the trajectory. Indeed, it may turn out, for example, that some specific η makes the denominator of the right-hand side of the equation vanish, whereas $\eta + 2\pi$ does not. When considering the motion in the real plane it would be inconvenient to have to distinguish different cases, according as the equation has or does not have a singular point. Changing to a Riemann surface eliminates this problem - the singular point lies on one sheet of the surface.

By introducing Riemann surfaces /3/, one can extend all the properties of the motion, as studied in /1, 4/, to the general case of arbitrary b.

1. Throughout the motion the angle η varies monotonically (and may leave the interval $[0, 2\pi]$.

2. If the initial value of η is a root of the function $f(\eta)$, it will maintain its value throughout the motion (both the relative motion and the absolute motion will be rectilinear) - this is the case of parallel convergence (the term is unfortunate, since the motion may also involve divergence).

3. If at the initial time $\eta \neq \eta_i$, where

$$f(\eta_i) = 0 \tag{2.1}$$

then the whole motion takes place in the sector between two roots of $f(\eta)$ (i.e., the relative trajectory cannot cross any of the rays $\eta = \eta_i$).

4. At the end of the motion always $\eta \to \eta_i$, and if at the same time $a \to 0$, then the motion will terminate in a finite time (barring the exceptional case in which $p = 1$ and the initial data are chosen in a very special way); but if a increases without limit or tends to a finite limit (the latter is possible only when $p = 1$), the motion will continue indefinitely.

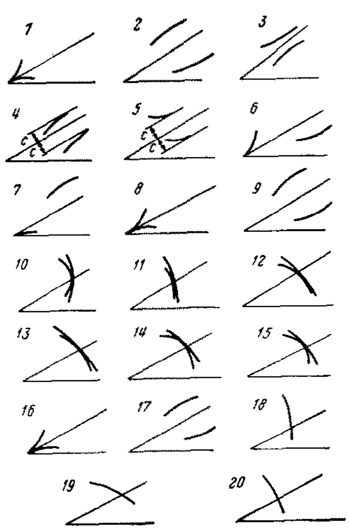

Fig.4

5. Near a ray $\eta = \eta_i$ the trajectory may have one of the 20 shapes shown in Fig.4. The shape of a trajectory is determined by the signs of the functions $L(\eta_i)$ and $\Omega(\eta_i)$ or of their derivatives, if the functions themselves vanish. (Note that some of the trajectories shown in Fig.4, such as Nos.1, 8, and 16 or Nos.2, 9, and 17, differ from each other only in the order of contact with the straight line $\eta = \eta_i$ or the rate of variation of the curvature near $\eta = \eta_i$.)

6. A qualitative picture of the shape of the trajectory in the sectors between the roots can be obtained by determining the roots of $F(\eta)$ and $\Omega(\eta)$ from the equations

$$F(\eta_{iF}) = 0,\quad \Omega(\eta_{i\Omega}) = 0 \tag{2.2}$$

constructing the rays $\eta = \eta_{iF}$ and $\eta = \eta_{i\Omega}$ (over the entire Riemann surface, or on that portion of the surface which is accessible from the first sheet if it is assumed that the initial data always correspond to the first sheet), with due allowance for the sign of the curvature (i.e., $\Omega(\eta)$), and the increase and decrease of the distance (i.e., the sign of $F(\eta)$) in the appropriate sectors (for examples of such constructions for $b = 3/4$, see /5/).

Of course, one can always assume that the initial data lie in the range $(0, 2\pi)$, but then the parameter ε_0 may lie outside $(0, 2\pi)$. If one starts instead with a prescribed value of ε_0, the values of η_0, ψ_0 must be chosen so that ε_0 takes this value. One can thus limit the range of ε_0.

7. Once the trajectories have been constructed the nature of the motion can be determined at once, if one takes into account that η varies monotonically and $\operatorname{sgn} \eta^{\cdot} = -\operatorname{sgn} f(\eta)$.

3. ***Stability of motion.*** Analysis of the stability of the motion in the classical sense requires a consideration of perturbations and the properties of the regulators. This was done for the linearized theory in /5/. It turns out that in the exact theory considered here this is less important, since the properties of the motion as determined are such that in most cases a comparatively slight action on the part of the regulators cannot exert a significant effect on them (of course, this does not eliminate the problems involved in allowing for lags and transients in the regulators, the effect on mismatch when the target is being approached, etc., all of which must be studied in addition).

We will therefore consider three aspects of stability.

Stability of the roots. Depending on whether $\eta \to \eta_i$ at the end of the motion, the roots of $f(\eta)$ may be classified as stable or unstable. Stable roots are those to which the angle η converges during the motion; unstable roots are those from which η diverges. There may also be semistable roots, which are limits of η from one side only. A classification of the roots according to the signs of the derivative $f'(\eta_i)$ (or of higher derivatives if $f'(\eta_i) = 0$) is given in Table 1. The table also classifies the roots as convergent or divergent, according to the sense in which the distance changes as the point moves along the ray $\eta = \eta_i$; if $p = 1$ there may also be neutral roots, at which the system is in a state of relative equilibrium (which may be either stable or unstable).

Table 1

No.	$\operatorname{sgn} f'(\eta_i)$	Change in sign of $f(\eta_i)$ on passing through the root	$\operatorname{sgn} F \times (\eta_i)$	Types of roots	
1	+1 or $f'(\eta_i)=f''(\eta_i)=0$, $f'''(\eta_i)>0$	$-$ to +	-1 +1 0	1) convergent 2) divergent 3) neutral	stable
2	-1 or $f'(\eta_i)=f''(\eta_i)=0$, $f'''(\eta_i)<0$	+ to $-$	-1 +1 0	4) convergent 5) divergent 6) neutral	unstable
3	0 $f''(\eta_i)\neq 0$	+ to + or $-$ to $-$	-1 +1 0	7) convergent 8) divergent 9) neutral	semi-stable

Stability of the trajectories. A trajectory is said to be stable if a small variation of the initial data produces a nearby trajectory. In this context there may be different definitions of "nearness". If "nearness" is measured in terms of angles, then since $\eta \to \eta_i$ all trajectories will be stable at the end of the motion, with the exception of rectilinear trajectories corresponding to unstable roots (for these trajectories a small perturbation will induce a finite change in the angle). If one considers instead the shortest distance between trajectories, then all trajectories for which the distance a tends to zero or to a finite limit are stable. As to trajectories for which $a \to \infty$, these will be stable in this sense only if the distance from the stright line $\eta = \eta_i$ tends to zero or to a finite limit (i.e., except for cases 2, 9, 17 and one of the sides in cases 6 and 7 in Fig.4), and only for perturbations not affecting the parameter ε_0. For perturbations that affect ε_0 (i.e., the roots η_i), trajectories going off to infinity are always unstable in this sense, since a small change in the roots when $a \to \infty$ will cause the change in distance to increase without limit. However, the most important stability concept in escape problems (in which one is interested primarily in increasing a) is obviously stability with respect to the angle (bearing), i.e., stability in the first sense.

Stability of convergence or divergence. By stability of convergence or divergence we mean that small variations in the initial data exert no influence on the final result of the motion.

This kind of stability is always present, unless the initial data pertain to a position at an unstable or semistable root. In the latter case a small perturbation (for a semistable root - a small perturbation displacing the point toward the unstable side) will cause a sudden change in the nature of the motion; the final result of the motion will be reversed: if the original root has a convergent one the point will move off; if it was divergent, the point will converge. These are the only cases in which a regulator exerting a small additional action (not comparable in magnitude with the force driving the basic motion (1.1)) may alter the end result of the motion. Such perturbations are obviously small variations of p or rotation of the polar axis through a small angle (i.e., manoeuvring of the point A), provided that the parameter values are not near the boundaries of the regions corresponding to each type of motion (see below).

4. Dependence of the motion on the parameters. As indicated in Sects.2 and 3, the nature of the motion is determined first and foremost by the values and types of the roots of the function $f(\eta)$, then by the roots of the functions $F(\eta)$ and $\Omega(\eta)$, and also by the relative positions of all these roots on the Riemann surface.

If one is not interested in the precise shape of the trajectories, it is sufficient to study the real roots of $f(\eta)$ and $F(\eta)$ and their relative positions. It follows from Eqs.(2.1) and (2.2) that the roots are functions of the parameters b, p, ε_0, and for fixed b they are functions of p and ε_0. Of fundamental importance in this connection is the curve in the p, ε_0 plane along which the equations have multiple roots, since it divides the plane into regions in which the function has the same number of roots, and each of these regions is only weakly dependent on the parameters. This curve is known as the separatrix (in the parameter plane) or discriminant curve. Both terms are also used in a somewhat different sense.

The parametric equations of the separatrix are

$$f(\eta) = \sin \eta + p \sin (b-1)(\eta - \varepsilon_0) = 0$$
$$f'(\eta) = \cos \eta + p(b-1) \cos (b-1)(\eta - \varepsilon_0) = 0 \tag{4.1}$$

Here η is a parameter (in fact - the value of a double root of Eq.(2.1)). The curve defined by Eq.(4.1) has two periods with respect to η:

$$T_1 = \begin{cases} \pi \,|(b-2)/(b-1)|, & b > 1 \\ \pi\, |b/(b-1)|, & b < 1 \end{cases} \tag{4.2}$$

is the period of a separate branch (and at the same time the period of each of the single-valued branches of the function $\eta_i(\varepsilon_0)$ defined for given p by Eq.(2.1)), and $T_2 = 2\pi / |b-1|$ is the period of the separatrix as a whole (corresponding to permutation of the roots of Eq.(2.1)).

If b is rational T_1 and T_2 are commensurable and the separatrix consists of a finite number of branches (this was proved for an integer in /1/). If b is irrational the separatrix will have an infinitesimal period, i.e., its periods form a module of the second kind (in the terminology of /6/). Similarly, one considers the separatrices of $F(\eta)$.

As we have stated, an important factor in determining the nature of the motion is the relative position of the roots of $f(\eta)$ and $F(\eta)$; hence it is also important to determine the mutual separatrix of the two functions, i.e., the curve in the parameter plane at whose points the functions have common roots. It follows from (2.1) and (2.2) that this is the straight line $p = 1$.

The separatrices of $f(\eta)$ and $F(\eta)$ lie entirely in the strip between the straight lines $p = 1$ and $p = |b-1|^{-1}$ (this was established in /1/ for integral $b \geqslant 3$). It follows that motions in the parameter regions above and below this strip have similar properties. Within the strip, study of the motion requires construction of the separatrices of $f(\eta)$ and $F(\eta)$, which divide the strip into certain subregions; one then determines which of these subregions contains the prescribed values of p and ε_0.

The above regions come together (i.e., the strip reduces to the straight line $p = 1$) only in case $b = 2$ (the case investigated in /7/). This once again emphasizes the exceptional nature of that case, implying that conclusions drawn for $b = 2$ cannot be carried over to the general case, as is still sometimes done erroneously.

General conclusions as to the nature of the motion are presented in Table 2 for the case $b < 1$, $b = 1 - q^{-1}$. The number of extrema on the trajectories is identical with the number of sectors of possible motion. Similar tables can be drawn up for rational b. The general conclusions drawn in these tables (except for the number of sheets of the Riemann surface) hold for any values of b in the appropriate intervals ($b < 1$, $1 < b < 2$, $b = 2$, $b > 2$). A slightly more accurate description of the shapes of the trajectories is achieved by considering the possibility that roots of $f(\eta)$ and $\Omega(\eta)$ might coincide. The corresponding curve in the parameter plane has the equation

$$\Omega(\eta_i) = 0 \tag{4.3}$$

where η_i are the roots of Eq.(2.1). This curve lies entirely in the region where $p \leqslant 1$, since it is obvious that if $p > 1$ the function $\Omega(\eta)$ has no roots. Eliminating η_i from (4.3) and (2.1), we obtain

$$p = \cos [\varepsilon_0 + {}^1/_2 (2m+1) \pi b (b-1)^{-1}] \tag{4.4}$$

As is evident from Table 2, the nature of the motion for the strip between $p = 1$ and $p = |b-1|^{-1}$ (this strip lies above the straight line $p = 1$ if $b < 2$, below it if $b > 2$) may be different.

To ascertain the nature of the motion it is in fact necessary to construct the separatrices of $f(\eta)$ and $F(\eta)$. To this end, in view of the periodicity of these functions, we need only construct one arc of the separatrix (the fundamental arc), with the parameter η varying from 0 to π. This arc is then repeated periodically, producing one branch of the separatrix, and the curve is then translated by multiplies of T_2. If b is rational this procedure yields only a fine number of branches, since then the periods T_1 and T_2 have some rational relation to one another, and a finite number of translations bring the branch back to its original position. If b is irrational, the curve will fill out the strip densely; but any

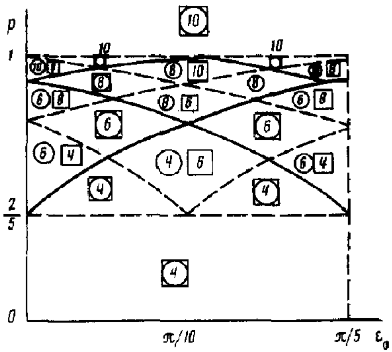

Fig.5

investigation of the dependence of the solution on the parameters must take into account only branches relating to the roots η lying on the appropriate sheets of the Riemann surface.

Fig.5 illustrates the partition of the parameter plane by the separatrices of $f(\eta)$ and $F(\eta)$ for $b = 7/2$. The solid curve represents the separatrix of $f(\eta)$, and the dashed curve that of $F(\eta)$. The figures in circles are the numbers of real roots of $f(\eta)$ (lying on two sheets of the Riemann surface, since this values of b has a denominator $q = 2$), those in squares are the numbers of roots of $F(\eta)$. For each of the subregions into which the separatrices divide the strip, the shape of the trajectories and the general nature of the motion are the same. Other examples may be found in /1/ (for $b = 3, 4$ and 5) and /6/ $b = 3/4$), where the reader will also find more detailed constructions of the trajectories for these cases. Note that the trajectories for $b = 7/2$ are intermediate between those for $b = 3$ and 4, considered in /1/, and it can indeed be shown that the types of motion in this case are intermediate between those exhibited in /1/.

5. ***The controllability of motion.*** In proportional navigation, motion is actually possible only if the normal accelerations created are not too large. The normal acceleration of a point is associated with the curvature of the absolute trajectory (briefly - the absolute curvature); in fact, as the magnitude of the velocity is constant, the acceleration can vary only due to a change in absolute curvature. An examination of Fig.1 will convince the reader that $w_n = K_a v^2 = \psi^{\cdot} v$, where w_n is the normal acceleration, K_a the absolute curvature, and $\psi^{\cdot}$ the angular velocity of the velocity vector of the absolute motion. Hence $K_a = \psi^{\cdot}/v$. Using Eqs.(1.4), and (1.6), we obtain

$$K_a = -bf(\eta) / (pa) \tag{5.1}$$

This equation enables one to treat the variation of the curvature along the trajectory as a function of the relative polar coordinate η. Differentiating (5.1) and using Eqs.(1.5) and (1.6), we obtain

$$\frac{dK_a}{d\eta} = \frac{1}{\eta^{\cdot}} \frac{dK_a}{dt} = -\frac{b}{p} \frac{\Phi(\eta)}{a} \tag{5.2}$$

$$\Phi(\eta) = f'(\eta) + F(\eta) = 2\cos(\eta) + p(b-2)\cos(b-1)(\eta - \varepsilon_0)$$

It is obvious that an extremum of the absolute curvature (hence, of the normal acceleration) may occur only at roots of $\Phi(\eta)$. If $\Phi(\eta)$ has no roots in the sector under consideration (i.e., in the sector between the initial value η_0 and the nearest stable root), or if the root corresponds to a minimum of $|K_a|$, then the maximum value of the curvature must occur at an endpoint of the interval. In that case one must consider $\lim K_a$ as $\eta \to \eta_i$, which can be done using the linarized theory, since the motion is taking place near a root.

In the linearized theory we have, near a simple root (see /1/),

$$a \approx c_1 |\eta - \eta_i|^{\alpha}, \quad \alpha = -F(\eta_i) / f'(\eta_i)$$

and so, for the absolute curvature near a root,

$$|K_a| \approx c_1 |\eta - \eta_i|^{r_i}, \quad r_1 = 1 + F(\eta_i)/f'(\eta_i) = \Phi(\eta_i)/f'(\eta_i) \tag{5.3}$$

Obviously, the behaviour of the curvature near a root is determined by the sign of r_i, and since in the case of a stable root (the only case of interest) $f'(\eta_i) > 0$, the sign of r_i is identical with that of $\Phi(\eta_i)$. Thus, the same function $\Phi(\eta)$ determines the variation of the curvature far from a root and the behaviour of the curvature near the root /8/.

If a stable root appears in a sector of the Riemann surface where $\Phi(\eta) > 0$, the curvature will tend to zero as that root is approached (Eq.(5.2) implies only that the curvature decreases in that sector). Consequently, by the terminology introduced in /1/, it is well controllable. Conversely, if $\Phi(\eta) < 0$ in a certain sector, then the curvature increases (by (5.2)) and tends to infinity (by (5.3)). It might appear that all this applies only in the case when the angle η increases. However, we are in fact interested not in K_a but in $|K_a|$; now, it follows from (5.1) and (5.2) that

$$\operatorname{sgn}(d|K_a| / d\eta) = \operatorname{sgn} f(\eta) \operatorname{sgn} \Phi(\eta) \tag{5.4}$$

and since it follows from (1.6) that $\operatorname{sgn} \eta^{\cdot} = -\operatorname{sgn} f(\eta)$, we see that if η increases with time we have the case already considered, whereas if η decreases with time the signs of $d|K_a|/d\eta$ and $\Phi(\eta)$ are the same.

Table 2

Value of p	No. of sectors of possible motion on Riemann surface	Possible no. of semi-stable roots	Variation of ν as function of η	Type of extrema	Sign of curvature of trajectory	Shape of trajectory at roots of $f(\eta)$ acc. of classification of Table 1	Possible types of roots of $f(\eta)$ in classification of Fig.4.	Relative position of roots of $f(\eta)$ and $F(\eta)$	Behaviour of distance a as the time increases
$p<1$	$2q$	None	Decreasing	Minima	Any	Types 2 & 4	Types 2, 3, also types 4, 5 on curve (4.5)	Separate each other	$\to\infty$
$p=1$	$2q$, except for values corresponding to minima of separatrix	One for even q, two for odd q	″	Minima (within interval)	+	Types 2 & 4 for roots distinct from roots of $F(\eta)$, types 7,8,9 for equal roots	Type 2 & types 11,13 at non-extrema of separatrices; type 15 possible at extremum of separatrix of $F(\eta)$ and type 17 at extremum of separatrix of $f(\eta)$	Some roots are identical	$\to$ or $\to A<\infty$
$\frac{1}{1-b}>p>1$	Intermediate between 2 and $2q$	One for even q, two for odd q (twice as many at points where the separatrix intersects itself)	May decrease and increase	both *maxima* and minima possible	+	Types 1,2, 4,5 outside separatrix, also types 3,6 on separatrix	Types 1,2 outside separatrix; possible cases on separatrix	May separate each other, but not in all sectors	$\to\infty$ or $\to 0$
$p=\frac{1}{1-b}$	2	None	Decreasing	Minima	+	Types 2&4	Type 2 at non-extrema of separatrix; type 9 possible at extrema of separatrix	Separate each other	$\to\infty$
$p>\frac{1}{1-b}$	2	None	Decreasing	Minima	+	Types 2&4	Type 2	Separate each other	$\to\infty$

Thus, once the rays $\eta = \eta_{i\Phi}$ corresponding to the roots of the equation $\Phi(\eta_{i\Phi}) = 0$, have been constructed, one can determine how the absolute curvature (i.e., normal acceleration) varies along a trajectory, and also determine its behaviour as a contact point is approached (if a contact point exists), i.e., as a stable convergent root is approached.

Having constructed the mutual separatrix of $f(\eta)$ and $\Phi(\eta)$, i.e., the curve defined parametrically by the equations

$$f(\eta) = \sin\eta + p\sin(b-1)(\eta-\varepsilon_0) = 0$$
$$\Phi(\eta) = 2\cos\eta + p(b-2)\cos(b-1)(\eta-\varepsilon_0) = 0$$

one can divide the parameter plane into regions, in each of which the roots are of the same type. the controllability of the roots may vary only on passing from one sector between roots of $\Phi(\eta)$ to another, i.e., when the functions $\Phi(\eta)$ and $f(\eta)$, have a common root; this principle, therefore, governs the dependence of the controllability on the parameters.

6. *Example*. Let us consider the case $b = {}^2/_3$, $p = 2$, $\varepsilon_0 = 0$ - these data will provide an adequate demonstration of the dependence of the stability and controllability on the parameter p. We have

$$f(\eta) = \sin\eta - p\sin{}^1/_3\,\eta,\quad F(\eta) = \cos\eta - p\cos{}^1/_3\eta$$
$$f'(\eta) = \cos\eta - {}^1/_3p\cos{}^1/_3\eta,\quad \Phi'(\eta) = 2\cos\eta - {}^4/_3p\cos{}^1/_3\eta$$

One of the roots of $f(\eta)$ is $\eta_1 = 0$.

We have $f'(\eta_1) = 1 - {}^1/_3p$. Thus, η_1 is a stable root if $p > 3$.

Since $F(\eta_1) = 1 - p$, η_1 is a convergent root if $p > 1$. However, the fact that it is both stable and convergent for $1 < p < 3$, does not imply that convergence from the region near this root actually occurs for all these values of p. Indeed, we have $\Phi(\eta_1) = 2 - {}^4/_3p$. Consequently, a root is well controllable only if $1 < p < {}^3/_2$ and poorly controllable if ${}^3/_2 < p < 3$. Hence it follows that at, say, $p = 2$, the motion cannot actually occur, since the normal acceleration in a motion approaching a contact point increases without limit. On the other hand, it is interesting that when $1 < p < {}^3/_2$ the normal acceleration on approaching a root tends to zero, i.e., contact is possible, despite the fact that the control is "weak" $(b - 1)$.

Note that the shape of the relative trajectory is not a clear indication of the controllability of roots. In the present example one can use a substitution generalizing that used in /2/ for integers b, to obtain the equation of the trajectory in a finite form.

If $Z = \mathrm{tg}{}^1/_3\eta$, then $g(\eta) = (1 + Z^2)^{3/2}\,|\,Z^2 - {}^1/_3\,|^{-3}\,|\,Z\,|^3$ gives the equation of the trajectory, apart from a costant factor.

The graph of the trajectory for $b = {}^2/_3$, $p = 2$, $\varepsilon_0 = 0$ (on an arbitrary scale) is shown in Fig.6. Since the relative distance a varies very strongly, one cannot illustrate the whole trajectory on the same scale. The figure therefore consists of five sections. Two of them (a and d) are drawn to a small scale, showing the general shape of the trajectory for certain sectors; the other three, drawn to a large scale, depict separate parts of the trajectory near the origin. These graphs illustrate the aforementioned phenomenon.

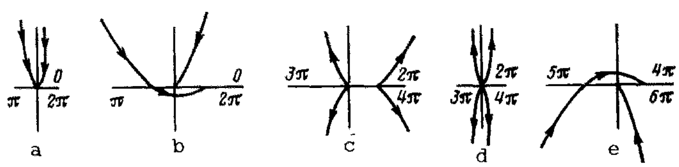

Fig.6

Indeed, when the point approaches, say, the ray $\eta = 2\pi$, it describes a trajectory quite different from its trajectory when approaching the ray corresponding to $\eta = 4\pi$, though the point of the plane is, geometrically speaking, exactly the same. The reason is the difference in conditions. Here the difference is automatically represented by the value of the angle η.

Convergence to the root η_1 is seen on the first and third sheets of the Riemann surface. In actual fact, however, as we have already shown, this convergence cannot actually occur.

It should be noted that the above example illustrates the influence of the parameter p. Since the initial data are partly incorporated in the parameter ε_0, while in other respects one is considering the entire trajectory, our example does not illustrate the effect of the initial data.

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OPTIMIZATION OF THE OBSERVATION PROCESS*

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The special problem of designing the observation process in cases when the observer parameters depend on the trajectory of the controlled dynamic system is considered. Such a dependence arises, for instance, when the measuring device is installed on a controlled moving platform (an aircraft) or if its parameters are affected by the dynamically varying characteristics of the environment (temperature). It is interesting to examine the selection of the trajectory of a dynamic system that minimizes the maximum possible estimation error (the size of the information set) /1, 2/. In formal terms, this question can be reduced to an optimal control problem with a non-smooth functional of a special form. Necessary conditions of optimality are given and some optimal observation processes are constructed.

Although the problem considered in this paper may be regarded as an infinite-dimensional generalization of some regression experiment design problem /3/, the results appear to be new and in a certain sense unexpected. Control of the size of the information set was previously considered in /4-6/.

1. Statement of the problem. The observed signal is given by

$$y(t) = a^*(t)\theta + \xi(t), \quad t \in [t_0, T] \tag{1.1}$$

where $\theta \in R^n$ is an unknown parameter vector, and $a(t) \in R^n$ is a known vector function whose components $a^i(\cdot)$ are assumed to be linearly independent and continuous on $[t_0, T]$; the unknown scalar disturbances $\xi(t)$ are bounded,

$$\langle \xi^2 \rangle \leqslant 1 \quad \left(\langle f \rangle = \int_{t_0}^{T} f(t)\,dt\right) \tag{1.2}$$

Here and henceforth, the asterisk denotes the transpose and $i = 1, 2, \ldots, n$.

For a fixed $y(\cdot)$ the set of vectors θ that satisfy (1.1), (1.2) is called an information set compatible with the realized signal /2/. In our case, the information set is an ellipsoid

$$E(\theta^\circ, P) = \{\theta \in R^n : (\theta - \theta^\circ)^* P (\theta - \theta^\circ) \leqslant 1 - h^2\} \tag{1.3}$$
$$\theta^\circ = P^{-1}d, \quad P = P(a(\cdot)) = \langle aa^* \rangle$$
$$d = \langle ay \rangle, \quad h^2 = \langle y^2 \rangle - d^* P^{-1} d$$

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